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Abstract—Discrete-time system inversion for perfect tracking goes at the expense of intersample behavior. The aim of this article is the development of a discrete-time inversion approach that improves continuous-time performance by also addressing the intersample behavior. The approach balances the on-sample and intersample behavior and provides a whole range of new solutions, with stable inversion and multirate inversion as special cases. The approach is successfully applied to a linear periodically time-varying system in both simulations and experiments. The approach improves the intersample behavior through discrete-time system inversion and outperforms existing approaches.

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Index Terms—Discrete-time inversion, intersample behavior, linear periodically time varying (LPTV), linear time invariant (LTI), multirate inversion, stable inversion.

I. INTRODUCTION

TRACKING control finds application in many areas, such as atomic force microscopes [1], wafer stages [2], and spectrometers [3]. The physical systems evolve in continuous time and, hence, their performance is naturally defined in continuous time. Many approaches for tracking control, including inverse model feedforward and iterative learning control, are based on system inversion. For continuous-time systems, system inversion approaches such as [4] can be used. However, controllers are often implemented in a digital environment since this provides a large flexibility at a low cost [5]. Due to the digital implementation, discrete-time control is often used.

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One of the main challenges in system inversion is nonminimum-phase behavior. Causal inversion of nonminimum-phase systems yields unbounded inputs. To avoid unbounded inputs, many discrete-time inversion approaches have been proposed, see, e.g., [6], for a recent overview. Approximate inversion approaches, such as zero-phase error tracking control [7], zero-magnitude error tracking control, and nonminimum-phase zero-ignore [8], are well known, but yield limited performance since an approximation is used. Optimal approaches, such as norm-optimal feedforward, \mathcal{H}_2 -preview control, and \mathcal{H}_{∞} -preview control [6, Sec. 4.3 and 4.4], yield high performance in discrete time. Discrete-time stable inversion [6, Sec. 4.2] yields exact tracking at the discrete-time samples.

Typically, discrete-time inversion approaches focus on the on-sample performance, i.e., at the discrete-time samples, resulting in poor intersample behavior, i.e., in between the samples, especially for zeros close to z = -1 [9]. This is observed for both linear time-invariant (LTI) and linear periodically time-varying (LPTV) systems [10]. As a consequence, the continuous-time behavior is poor. Indeed, the best on-sample performance does not necessarily leads to the best continuous-time performance.

Multirate inversion [11], [12] provides an interesting alternative to improve intersample behavior by sacrificing on-sample performance. However, the approach does not take into account the system dynamics when balancing the intersample and onsample performance. As a consequence, the continuous-time performance will in general be suboptimal.

Although there exist many discrete-time inversion approaches, the balance between on-sample performance and intersample behavior is often not addressed. The main contribution of this article is a discrete-time inversion approach that finds the optimal balance between on-sample performance and intersample behavior for the purpose of continuous-time performance, for both LTI and LPTV systems. LPTV systems are of interest since they occur frequently, including sampled-data systems [5]; multirate systems [5], [13]; position-dependent systems with periodic tasks [14]; and nonequidistant sampling [15]. For both LTI and LPTV systems, the stable inversion and multirate inversion approaches are recovered as special cases. Related work includes [5], [16], and [17] where synthesis-based approaches are presented. The approach presented in this article does not require synthesis.

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Fig. 1. Tracking control diagram with continuous-time system H_c , sampler S, and hold \mathcal{H} . Given continuous-time reference trajectory r(t), the objective is to minimize continuous-time error e(t) through design of discrete-time controller F while control input u[k] remains bounded.

The outline of this article is as follows. In Section II, the control diagram is presented and the control objective is formulated. The main idea of the approach and preliminary results are presented in Section III. The approach is presented in Section IV. The advantages of the approach are demonstrated by application to an LPTV motion system in simulations and experiments in Section V. Section VI concludes this article.

Notation: For notational convenience, single-input, singleoutput systems are considered. The results can directly be generalized to square multivariable systems. Let $s^{(i)} \triangleq \frac{d^i}{dt^i}s$ denote the *i*th time-derivative of *s*, $\mathcal{B}(\cdot)$ a bilinear transformation, and $\mathbb{R}^{b}_{>a} = \{x \in \mathbb{R}^{b} | x[k] > a$ for all $k = 0, 1, \ldots, b-1\}$. Let $\Sigma \stackrel{z}{=} (A, B, C, D)$ be a discretetime state-space model and define the state transformation $\mathcal{T}(\Sigma, T) \stackrel{z}{=} (TAT^{-1}, TB, CT^{-1}, D)$.

II. PROBLEM FORMULATION

In this section, the control problem is formulated. The considered tracking control configuration is shown in Fig. 1, with reference trajectory $r(t) \in \mathbb{R}$, control input $u(t) \in \mathbb{R}$, output $y(t) \in \mathbb{R}$, digital controller F, sampler S, and zero-order hold \mathcal{H} . The continuous-time, LTI system H_c is given by the minimal realization

$$\dot{x}(t) = A_c x(t) + B_c u(t) \tag{1a}$$

$$y(t) = C_c x(t) \tag{1b}$$

with $x(t) \in \mathbb{R}^n$, $n \in \mathbb{N}$ and can be either an open-loop or closed-loop system. It is assumed that H_c is stable.

In conventional discrete-time control, the focus is on onsample performance. The discrete-time system $H = SH_cH$ with H_c in (1) and sampling time δ is given by

$$x[k+1] = Ax[k] + Bu[k]$$
(2a)

$$y[k] = Cx[k] \tag{2b}$$

with

$$A = e^{A_c \delta}, \qquad B = \int_0^\delta e^{A_c \tau} B_c \,\mathrm{d}\tau, \qquad C = C_c. \tag{2c}$$

In this setting, perfect on-sample tracking, i.e., e[k] = 0, for all k, is achieved for $F = H^{-1}$. However, this does not provide any guarantees for the intersample performance e(t), $t \neq k\delta$. Hence, the continuous-time performance in terms of e(t), for all t, may be poor as observed in, e.g., [10].

The control objective considered in this article is to minimize the continuous-time error e(t). Note that this includes



Fig. 2. Block diagram of the approach. The discrete-time system H is decomposed into H_1 and H_2 . System H_1 is inverted such that there is exact state tracking of the desired state $\hat{x}_1[k]$ every n_1 samples for the purpose of intersample behavior. System H_2 is inverted such that there is exact output tracking every sample for the purpose of on-sample behavior.

both on-sample ($t = k\delta$) and intersample ($t \neq k\delta$) performance. Importantly, u[k] should remain bounded, even in the presence of nonminimum-phase behavior. Trajectory r(t) is assumed to be known *a priori*.

In the next section, the main idea of the approach and preliminary results are presented.

III. CONCEPTUAL IDEA AND PRELIMINARY RESULTS

In this section, the conceptual idea of the approach and preliminary results are presented. The results form the basis for the complete approach presented in Section IV.

A. Conceptual Idea

In the proposed approach, the system is decomposed into two parts and both parts are inverted separately according to Fig. 2, where H is decomposed as $H = H_1H_2$. The inversion of system H_1 aims at the intersample behavior. More specific, let n_1 be the state dimension of H_1 , then H_1 is inverted such that there is exact state tracking of a desired state $\hat{x}_1[k]$ every n_1 samples. The inversion of H_2 aims at the on-sample behavior through perfect output tracking for every sample.

Exact state tracking is experienced to yield good intersample behavior in multirate inversion [11], whereas exact output tracking yields good on-sample behavior in stable inversion [6]. Hence, the choice of the decomposition into H_1 and H_2 can be used to balance the on-sample behavior and the intersample behavior to the benefit of the continuous-time performance. The idea is conceptually illustrated in Fig. 3. An important observation is that a small on-sample error does not necessarily yield a small continuous-time error. The figure shows that the approach provides a whole range of solutions that were nonexisting before. The stable inversion and multirate inversion solution are recovered as the two extreme cases, see also Section IV-C.

The approach requires the decomposition $H = H_1H_2$ in terms of state-space realizations and the desired state $\hat{x}_1[k]$ for H_1 , see also Fig. 2. In Section III-B, the desired state for the continuous-time system H_c is presented. In Section III-C, the state-space decomposition $H = H_1H_2$ is presented. The results form the basis for the complete approach presented in Section IV.



Fig. 3. Qualitative plot of the continuous-time error versus the onsample error. The approach balances the intersample behavior and the on-sample behavior for the purpose of continuous-time performance. It provides a whole range of solutions (**O**) that were nonexisting before. Importantly, the smallest on-sample error does not necessarily yield the smallest continuous-time error. The relative performance depends on the particular settings, e.g., the system dynamics, and may vary. Stable inversion (**■**) and multirate inversion (�) are recovered as special cases.

B. Desired State for Continuous-Time System

In this section, the desired state for the continuous-time system is presented. Given a continuous-time reference trajectory r(t) together with its n-1 time derivatives and system H_c in (1), the objective is to determine a bounded state $\hat{x}(t)$ such that $y(t) = C_c \hat{x}(t)$ yields $y^{(i)}(t) = r^{(i)}(t)$, $i = 0, 1, \ldots, n-1$, where $(\cdot)^{(i)}$ denotes the *i*th time derivative of (\cdot) , i.e., such that $\bar{r}(t) = \bar{y}(t)$ where

$$\bar{r}(t) = \begin{bmatrix} r^{(0)}(t) \\ r^{(1)}(t) \\ \vdots \\ r^{(n-1)}(t) \end{bmatrix}, \quad \bar{y}(t) = \begin{bmatrix} y^{(0)}(t) \\ y^{(1)}(t) \\ \vdots \\ y^{(n-1)}(t) \end{bmatrix}. \quad (3)$$

A similar approach as in [11] is used based on the controllable canonical form given by Lemma 1, see also [18, Sec. 17.6]. The desired state is given by Theorem 2. The results are recapitulated for completeness and reading convenience.

Lemma 1. (Controllable canonical form): Let the transfer function of H_c in (1) be given by

$$H_{c} = C_{c}(sI - A_{c})^{-1}B_{c} = \frac{B(s)}{A(s)}$$
(4a)

with

$$A(s) = \frac{s^n + a_{n-1}s^{n-1} + \dots + a_0}{b_0}$$
(4b)

$$B(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_0}{b_0}$$
(4c)

 $b_0 \neq 0$, then the controllable canonical form $H_{ccf} = \mathcal{T}(H_c, T_{ccf})$ is given by

$$\dot{x}_{ccf}(t) = A_{ccf} x_{ccf}(t) + B_{ccf} u(t)$$
(5a)

$$y(t) = C_{ccf} x_{ccf}(t) \tag{5b}$$

where

$$\begin{bmatrix} \underline{A_{ccf} | B_{ccf}} \\ \overline{C_{ccf} |} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \underline{-a_0 - a_1 - a_2 \cdots - a_{n-1} b_0} \\ \hline 1 & \frac{b_1}{b_0} & \frac{b_2}{b_0} & \cdots & 0 \end{bmatrix}$$
(5c)

and

$$T_{ccf}^{-1} = \begin{bmatrix} B_c & A_c B_c & \cdots & A_c^{n-1} B_c \end{bmatrix} \begin{bmatrix} \frac{a_1}{b_0} & \frac{a_2}{b_0} & \cdots & \frac{1}{b_0} \\ \frac{a_2}{b_0} & \frac{a_3}{b_0} & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \frac{1}{b_0} & 0 & \cdots & 0 \end{bmatrix} .$$
(6)

Theorem 2. (Desired continuous-time state): Let $B^{-1}(s)$ in (4) be decomposed as

$$B^{-1}(s) = F_s(s) + F_u(s)$$
(7)

with all poles $p_s \in \mathbb{C}$ of $F_s(s)$ such that $\Re(p_s) < 0$ and all poles $p_u \in \mathbb{C}$ of $F_u(s)$ such that $\Re(p_u) > 0$. Let

$$f_s(t) = \mathcal{L}^{-1}(F_s(s)), \qquad f_u(t) = \mathcal{L}^{-1}(F_u(-s))$$
 (8a)

$$\hat{x}_{ccf,s}(t) = \int_{-\infty}^{t} f_s(t-\tau)\bar{r}(\tau) \,\mathrm{d}\tau \tag{8b}$$

$$\hat{x}_{ccf,u}(t) = \int_{t}^{\infty} f_u(t-\tau)\bar{r}(\tau) \,\mathrm{d}\tau$$
(8c)

where $\mathcal{L}^{-1}(\cdot)$ is the inverse unilateral Laplace transform [19, Sec. 9.3]. Let H_c in (4) have realization (5), then $y(t) = C_c \hat{x}(t)$ where

$$\hat{x}(t) = T_{ccf}^{-1}(\hat{x}_{ccf,s}(t) + \hat{x}_{ccf,u}(t))$$
(9)

is bounded and such that $\bar{y}(t) = \bar{r}(t)$, with $\bar{y}(t)$ and $\bar{r}(t)$ in (3). *Proof:* See [11, Sec. IV-B].

Theorem 2 provides the desired bounded state for optimal state tracking. Together with the state-space decomposition presented in the next section, Theorem 2 forms the basis of the approach presented in Section IV.

Remark 1: If poles of $B^{-1}(s)$ in Theorem 2 have $\Re(p) = 0$, i.e., $B^{-1}(s)$ is nonhyperbolic, similar techniques as in [20] can be used.

C. State-Space Decomposition

In this section, the multiplicative state-space decomposition is presented. A multiplicative decomposition, in contrast to an additive decomposition, enables exact on-sample tracking every few samples. Together with Theorem 2, the decomposition forms the basis of the approach in Section IV.

Given the state-space system H in (2), the interest is in minimal realizations H_1, H_2 such that $H = H_1H_2$ in terms of state-space realization, where the zeros and poles of H can be arbitrarily assigned to H_1 or H_2 . The starting point is the multiplicative decomposition $H = H_1H_2$ in terms of transfer functions as given by Lemma 3. Lemma 3. (Transfer function decomposition): Let $H \stackrel{z}{=} (A, B, C, D)$ be a state-space realization with n states and invertible D. Let $V \in \mathbb{R}^{n \times n_1}$ be a column space of an invariant subspace of A and let $V_{\times} \in \mathbb{R}^{n \times n_2}$ be a column space of an invariant subspace of $A_{\times} = A - BD^{-1}C$, such that $S = \begin{bmatrix} V & V_{\times} \end{bmatrix}$ has full rank $n = n_1 + n_2$. Let

$$\Pi = S \begin{bmatrix} I_{n_1} & 0_{n_1 \times n_2} \\ 0_{n_2 \times n_1} & 0_{n_2 \times n_2} \end{bmatrix} S^{-1}.$$
 (10)

Then, the realizations

$$H_{1f} \stackrel{z}{=} \left[\frac{A|\Pi BD^{-1}}{C|I}\right], \ H_{2f} \stackrel{z}{=} \left[\frac{A|B}{C(I-\Pi)|D}\right]$$
(11)

are such that $H = H_{1f}H_{2f}$ in terms of transfer functions, i.e., $C(zI - A)^{-1}B + D = (C(zI - A)^{-1}\Pi BD^{-1} + I)(C(I - \Pi)(zI - A)^{-1}B + D).$

Proof: Follows directly from extending [21, Corrollary 11] to $D \neq I$.

If the *D* matrix in Lemma 3 is singular, a bilinear transformation [5, Sec. 3.4], [19, Sec. 10.8.3] can possibly be employed to obtain an equivalent system with nonsingular *D* matrix. A multiplicative decomposition for the transformed system is obtained through Lemma 3. Applying the inverse transformation on the decomposed system yields the decomposition for the original system since $\mathcal{B}(H_1H_2) = \mathcal{B}(H_1)\mathcal{B}(H_2)$.

Importantly, Lemma 3 guarantees equivalence in terms of transfer functions, but not in terms of state-space realizations. Indeed, the decomposition of Lemma 3 yields nonminimal realizations of H_{1f} , H_{2f} as both have state dimension n. By exploiting the modal form and using a suitable state transformation, the desired state-space decomposition for the approach is obtained as given by Theorem 4.

Theorem 4. (State-space decomposition): Let $T_{\text{mod}} \in \mathbb{C}^{n \times n}$ be such that $H_{\text{mod}} = \mathcal{T}(H, T_{\text{mod}}) = (A, B, C, D)$ is in modal form [22, Sec. 7.4] with nonsingular D. Let $H_{1f}H_{2f} = H_{\text{mod}}$ be the decomposition given by Lemma 3. Let $T_{\text{per}} \in \mathbb{R}^{n \times n}$ be such that

$$\mathcal{T}(H_{1f}, T_{\text{per}}) \stackrel{z}{=} \begin{bmatrix} A_1 & 0 & B_1 \\ 0 & A_2 & 0 \\ \hline C_1 & C_{1r} & I \end{bmatrix}$$
(12)

$$\mathcal{T}(H_{2f}, T_{\text{per}}) \stackrel{z}{=} \begin{bmatrix} A_1 & 0 & B_{2r} \\ 0 & A_2 & B_2 \\ \hline 0 & C_2 & D \end{bmatrix}$$
(13)

with $A_1 \in \mathbb{R}^{n_1 \times n_1}$, $A_2 \in \mathbb{R}^{n_2 \times n_2}$, $n_1 + n_2 = n$, and define

$$H_1 \stackrel{z}{=} \left[\frac{A_1 \mid B_1}{C_1 \mid I} \right], \qquad H_2 \stackrel{z}{=} \left[\frac{A_2 \mid B_2}{C_2 \mid D} \right]. \tag{14}$$

Furthermore, let $X \in \mathbb{R}^{n_1 \times n_2}$ satisfy

$$A_1 X - X A_2 = B_1 C_2. (15)$$

Then, the state-space realization of $\mathcal{T}(H_1H_2, T_{per}^{-1}T_{12})$ with

$$T_{12} = \begin{bmatrix} I_{n_1} & X\\ 0_{n_1 \times n_2} & I_{n_2} \end{bmatrix}$$
(16)

is identical to that of H_{mod} .

Proof: See Appendix A.

Theorem 4 yields a state-space decomposition $H = H_1H_2$ with identical state-space realizations. Note that such a decomposition always exists. Together with Theorem 2, Theorem 4 forms the basis for the approach presented in the next section.

Remark 2: Note that V in Lemma 3 is related to the poles of H, whereas V_{\times} is related to the zeros of H. Hence, V, V_{\times} can be used to assign the poles and zeros to either H_1 or H_2 .

Remark 3: The column spaces of the invariant subspaces in Lemma 3 can be constructed from eigenvectors. Note that for complex eigenvectors, the real and imaginary parts should be used. For eigenvalues with multiplicity larger than one, generalized eigenvectors obtained from the Jordan form can be used to ensure *S* has full rank.

Remark 4: Sylvester (15) has a unique solution X if the eigenvalues of A_1 and $-A_2$ are distinct [23].

IV. INVERSION FOR INTERSAMPLE PERFORMANCE

In the previous section, the global idea and preliminary results on the desired state and the state-space decomposition are presented. Based on these results, the approach is presented. First, the approach for LTI systems is presented. Second, the approach for LPTV systems is presented. Finally, special cases are recovered.

A. Approach for LTI Systems

The approach consists of two steps. First, stable inversion is applied to H_2 in (14) to obtain u[k] such that $y_2[k] = u_1[k]$, for all k, see also Fig. 2. The solution is given by Theorem 5 and provides exact output tracking every sample. See [6, Sec. 4.2] for a proof.

Theorem 5. (Inversion of H_2): Consider Fig. 2 and let H_2^{-1} be given by

$$\begin{bmatrix} x_s[k+1] \\ x_u[k+1] \end{bmatrix} = \begin{bmatrix} A_s & 0 \\ 0 & A_u \end{bmatrix} \begin{bmatrix} x_s[k] \\ x_u[k] \end{bmatrix} + \begin{bmatrix} B_s \\ B_u \end{bmatrix} u_1[k] \quad (17a)$$

$$u[k] = \begin{bmatrix} C_s & C_u \end{bmatrix} \begin{bmatrix} x_s[k] \\ x_u[k] \end{bmatrix} + Du_1[k]$$
(17b)

with $|\lambda(A_s)| < 1$ and $|\lambda(A_u)| > 1$. Then, $y_2[k] = u_1[k]$, for all k, if

$$u[k] = C_s x_s[k] + C_u x_u[k] + Du_1[k]$$
(18)

which is bounded for bounded u_1 and where x_s follows from solving

$$x_s[k+1] = A_s x_s[k] + B_s u_1[k], \quad x_s[-\infty] = 0$$
(19)

forward in time and x_u follows from solving

$$x_u[k+1] = A_u x_u[k] + B_u u_1[k], \quad x_u[\infty] = 0$$
(20)

backward in time.

If $u_1[k]$ is bounded, u[k] in Theorem 5 is bounded by construction of $x_s[k], x_u[k]$, even if H_2 is nonminimum phase. The stable inversion solution in Theorem 5 achieves exact output tracking every sample and has infinite preactuation. Regular

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causal inversion is recovered as special case if the system is minimum phase (x_u is nonexisting), see also [6].

Second, multirate inversion is applied to H_1 in (14) to obtain $u_1[k]$. Note that by Theorem 5, $y_2[k] = u_1[k]$, for all k. The solution is based on lifting the state equation over n_1 samples. The solution is adopted from [11] and given by Theorem 6. The solution provides exact state tracking every n_1 samples.

Theorem 6. (Inversion of H_1): Consider Fig. 2 with $y_2[k] = u_1[k]$, for all k, and let $\hat{x}_1[k]$ be the desired state for system H_1 in (14). Consider the state equation lifted over τ samples given by

$$\underline{x}_1[q+1] = \underline{A}_1 \, \underline{x}_1[q] + \underline{B}_1 \, \underline{u}_1[q] \tag{21}$$

with $\underline{A}_1 = A_1^{n_1}$, $\underline{B}_1 = [A_1^{n_1-1}B_1 \ A_1^{n_1-2}B_1 \ \dots \ B_1]$, $\underline{u}_1[q] = [u_1[kn_1] \ u_1[kn_1+1] \ \dots \ u_1[(k+1)n_1-1]]^{\top}$, and $\underline{x}_1[q] = x_1[kn_1]$. Then, $\underline{x}_1[q] = \underline{\hat{x}}_1[q]$, for all q, where $\underline{\hat{x}}_1[q]$ is $\hat{x}_1[k]$ lifted over τ samples, if

$$\underline{u}_1[q] = \underline{B}_1^{-1} \left(\underline{\hat{x}}_1[q+1] - \underline{A}_1 \, \underline{\hat{x}}_1[q] \right) \tag{22}$$

which is bounded for bounded \hat{x}_1 .

Proof: See [11, Sec. IV-C].

Importantly, the inversion approach in Theorem 6 is based on the continuous-time system H_c , rather than the discrete-time system H. The approach yields exact state tracking, and, hence, exact output tracking, every n_1 samples and has n_1 samples preactuation. Note that u_1 is bounded if \hat{x}_1 is bounded, even if H_1 is nonminimum phase. More details can be found in, for example, [11] and [12]. The desired state \hat{x}_1 in Theorem 6 is obtained by Procedure 7, which follows from Sections III-B and III-C. An example of the application of Procedure 7 to a motion system is provided in Appendix B.

Procedure 7. (Desired state of H_1): Given H_c in (5), H in (2), and the decomposition $H = H_1H_2$ in Theorem 4, the following steps yield the desired state $\hat{x}_1[k]$ in Theorem 6.

- 1) Obtain the controllable canonical form $H_{ccf} = \mathcal{T}(H_c, T_{ccf})$ according to Lemma 1.
- 2) Obtain the desired state $\hat{x}(t)$ of H_c using Theorem 2.
- 3) Set the desired state of *H* to $\hat{x}[k] = \hat{x}(k\delta)$.
- 4) Obtain the desired state of H_{mod}: x̂_{mod}[k] = T_{mod}x̂[k], with H_{mod}, T_{mod} in Theorem 4.
- 5) Given H_1, H_2 in (14), let

$$H_{12} = H_1 H_2 \stackrel{z}{=} \begin{bmatrix} A_1 & B_1 C_2 & B_1 D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1 C_2 & D_1 D_2 \end{bmatrix}.$$
 (23)

- 6) Obtain the desired state of H_{12} : $\hat{x}_{12}[k] = T_{12}^{-1}T_{per}$ $\hat{x}_{mod}[k]$, with T_{12} in (16) and T_{per} satisfying (12) and (13).
- 7) Obtain the desired state for H_1 : $\hat{x}_1[k] = [I_{n_1} \quad 0_{n_1 \times n_2}]\hat{x}_{12}[k].$

The combination of the inversion of H_2 in Theorem 5 and the inversion of H_1 in Theorem 6 constitutes the control input u[k] in Fig. 2, which is bounded by design, also for nonminimum-phase systems. The design freedom is in the decomposition of H into H_1 and H_2 in Theorem 4. Equation (23) shows that the output is given by $y[k] = C_1 x_1[k] + D_1 C_2 x_2[k]$ since either $D_1 = 0$ or $D_2 = 0$ as D = 0 in (2). If $D_1 = 0$, $y[k] = C_1 x_1[k]$ and since inversion of H_1 in Theorem 5 ensures perfect state tracking of $x_1[k]$ every n_1 samples, there is perfect output tracking every n_1 samples. If $D_1 \neq 0$, y[k] also depends on $x_2[k]$ of H_2 and since inversion of H_2 in Theorem 6 does not provide perfect state tracking, there is no perfect output tracking for y[k] every n_1 samples. Hence, to guarantee exact on-sample tracking every n_1 samples, V, V_{\times} in Theorem 4 are preferably chosen such that $D_1 = 0$.

In this section, the approach for LTI systems is presented. In the next section, the approach for LPTV systems is presented.

Remark 5: For strictly proper systems H_2 , Theorem 5 can be applied to the biproper system \bar{H}_2 obtained through time shifts $\bar{H}_2 = z^{d_2}H_2$, where d_2 is the relative degree of H_2 , see also [6, Remark 1]. If there are eigenvalues on the unit circle, i.e., there exist λ_i such that $|\lambda_i(A)| = 1$, then similar techniques as in [20] can be followed.

Remark 6: The decomposition of H^{-1} given by (17) can be obtained through an eigenvalue decomposition.

Remark 7: Note that \underline{B}_1 in Theorem 6 is the controllability matrix of H_1 , and hence, \underline{B}_1^{-1} exists if H_1 is controllable.

B. Approach for LPTV Systems

In this section, the approach for LPTV systems is presented. Let the LPTV system H with period $\tau \in \mathbb{N}$ be given by

$$x[k+1] = A[k]x[k] + B[k]u[k]$$
(24a)

$$y[k] = C[k]x[k] \tag{24b}$$

with $A[k + \tau] = A[k]$, $B[k + \tau] = B[k]$, and $C[k + \tau] = C[k]$, for all k. LPTV systems may result from nonequidistant sampling as in Example 1.

Example 1. (Nonequidistant sampling): Let the sampling in Fig. 1 be nonequidistant in time and given by the sampling sequence $\Delta_{ne} \in \mathbb{R}_{>0}^{\infty}$ with periodicity $\tau \in \mathbb{N}$ defined as

$$\Delta_{ne} = (\delta_1, \delta_2, \dots, \delta_\tau, \delta_1, \delta_2 \dots) \tag{25}$$

with $\delta_i = \gamma_i \delta_b$, $\delta_b \in \mathbb{R}_{>0}$, $\gamma_i \in \mathbb{N}$, $i = 1, 2, ..., \tau$. Then, the discretized system $H = SH_c \mathcal{H}$ is given by (24) with

$$A[i] = e^{A_c \delta_i}, \quad B[i] = \int_0^{\delta_i} e^{A_c \tau} B_c \, \mathrm{d}\tau, \quad C = C_c \qquad (26a)$$

 $i = 1, 2, ..., \tau$, where $A[k + \tau] = A[k]$ and $B[k + \tau] = B[k]$, for all k. By linearity of H_c and periodicity of Δ_{ne} , H is LPTV with period τ .

The approach for LPTV systems is similar to that for LTI systems, with the key difference that an additional lifting step is used. The lifting step turns the LPTV system into a (multivariable) LTI system as given by Lemma 8.

Lemma 8: Lifting the input of H in (24) over τ samples yields the LTI system \underline{H} given by

$$\underline{x}[q+1] = \underline{A}\,\underline{x}[q] + \underline{B}\,\underline{u}[q] \tag{27a}$$

$$y[q] = \underline{C} \underline{x}[q] + \underline{D} \underline{u}[q]$$
(27b)

where

$$\underline{x}[q] = x[k\tau], \quad \underline{u}[q] = \begin{bmatrix} u[k\tau] \\ u[k\tau+1] \\ \vdots \\ u[(k+1)\tau-1] \end{bmatrix}$$
(27c)
$$\begin{bmatrix} \underline{A} \mid \underline{B} \\ \overline{C} \mid \overline{D} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{\Phi_{\tau+1,1} \mid \Phi_{\tau+1,2}B[1] \mid \Phi_{\tau+1,3}B[2] \dots B[\tau]}{C[1] \mid 0 \quad 0 \quad \cdots \quad 0} \\ C[2]\Phi_{2,1} \mid C[2]B[1] \mid 0 \quad \cdots \quad 0 \\ \vdots \quad \vdots \quad \vdots \quad \ddots \quad \vdots \\ C[\tau]\Phi_{\tau,1} \mid C[\tau]\Phi_{\tau,2}B[1] \mid C[\tau]\Phi_{\tau,3}B[2] \cdots \quad 0 \end{bmatrix}$$
(27d)

with transition matrix

$$\Phi_{k_2,k_1} = \begin{cases} I, & k_2 = k_1 \\ A[k_2 - 1]A[k_2 - 2] \dots A[k_1], & k_2 > k_1. \end{cases}$$
(28)

For the lifted system \underline{H} in (27), the same approach as for the LTI system illustrated in Fig. 2 is used. The state-space decomposition $\underline{H} = \underline{H}_1 \underline{H}_2$ is obtained using Theorem 4. System \underline{H}_2 is inverted using Theorem 5 and \underline{H}_1 is inverted using Theorem 6, where the desired state $\hat{\underline{x}}_1[q]$ follows along the same lines as in Procedure 7. The result is the lifted input signal $\underline{u}[q]$, which, after inverse lifting, yields input u[k], for the LPTV system H in (24).

In the previous and present section, the approaches for LTI and LPTV are presented, respectively. Next, special cases are recovered.

C. Special Cases

The approach provides a whole range of solutions as illustrated in Fig. 3. The stable inversion and multirate inversion solution are recovered as the two extreme cases and given by Corollaries 9 and 10. The results hold for both LTI and LPTV systems.

Corollary 9. (Special case: Stable inversion [6], [24]): The stable inversion solution for H is recovered from the approach in Section IV as special case if $H = H_2$, i.e., $H_1 = I$ and $n_1 = 0$.

Corollary 10. (Special case: Multirate inversion [11], [12]): The multirate inversion solution for H is recovered from the approach in Section IV as special case if $H = H_1$, i.e., $H_2 = I$ and $n_1 = n$.

Importantly, although Theorem 5 yields exact output tracking of H_2 for every sample, the inversion of H_1 does not reduce to conventional multirate inversion of H_1 since the desired state $\hat{x}_1[k]$ depends on the full system H_c and not only on H_1 .

Finally, the approach for LTI systems is recovered from that for LPTV systems as given by Corollary 11. Indeed, for $\tau = 1$, (2) is recovered from (24).

Corollary 11: The approach is applicable to both LTI and LPTV systems. Indeed, the approach for LTI systems in



Fig. 4. Motion system [25] used in simulations and for experimental validation. (a) Experimental high-precision positioning stage with force input u and output displacement y. (b) Bode diagram of a frequency response function measurement (•) of the motion system in Fig. 4(a) and the identified continuous-time model H_c (----).

Section IV-A is recovered as a special case from the approach for LPTV systems in Section IV-B for $\tau = 1$.

The approach provides a whole range of solutions that were nonexisting before, see also Fig. 3. The advantages of the approach are demonstrated by application to an LPTV motion system in Section V.

V. EXPERIMENTAL RESULTS

In this section, the approach outlined in Section IV is validated experimentally on a motion system. The experiments show the applicability of the approach in a practical setting. First, the motion system is presented. Second, the approach is validated in simulations revealing the desired properties and providing the optimal decomposition for use in the experiments. Finally, the approach is validated in experiments. Both in simulations and in experiments, the approach is superior to the special cases of stable inversion and multirate inversion.

A. Motion System

The approach is validated on the experimental high-precision positioning stage shown in Fig. 4(a). The Bode diagram of a frequency response function measurement of the system is shown in Fig. 4(b). The measurement is obtained through a dedicated identification experiment with a multisine input in the



Fig. 5. Reference trajectory r(t) consisting of eighth-order polynomials in time.

range between 1–1000 Hz and a sampling frequency of 12.5 kHz. The identified eighth-order continuous-time system H_c (n = 8 and m = 4) is given by

$$H_{c} = \frac{3.7232 \cdot 10^{6} (s^{2} + 7.181s + 2.507 \cdot 10^{4})}{s(s + 2.33)(s^{2} + 9.132s + 3.672 \cdot 10^{4})} \\ \times \frac{(s^{2} + 102.6s + 8.531 \cdot 10^{5})}{(s^{2} + 37.91s + 3.12 \cdot 10^{5})(s^{2} + 254.5s + 3.478 \cdot 10^{6})}$$
(29)

and is stable and minimum phase. The Bode diagram of the model H_c is also shown in Fig. 4(b). The continuous-time reference trajectory r(t) is shown in Fig. 5.

A nonequidistant sampling sequence with $\gamma_1 = 1$, $\gamma_2 = 2$, and $\delta_b = 400 \ \mu s$ ($f_b = 2.5 \ \text{kHz}$) is used, see also Example 1, resulting in an LPTV system H with period $\tau = 2$. The corresponding lifted LTI system \underline{H} in (27) has one nonminimumphase (transmission) zero due to discretization.

B. Simulation: Finding the Optimal Decomposition

In this section, the approach in Section IV is evaluated in simulation to confirm its properties and to find the optimal decomposition for use in the experiments. In Section V-C, the approach is validated in experiments.

First, three different solutions are considered: the approach in Section IV with $n_1 = 4$, the special case $n_1 = 0$, i.e., multirate inversion in Corollary 10, and the special case $n_1 = 8$, i.e., stable inversion in Corollary 9. The input signals are shown in Fig. 6. Note that the input signal of stable inversion is nonsmooth. The error signals are shown in Fig. 7, which, for the purpose of intersample performance evaluation, are evaluated at a sampling frequency of 250 kHz, i.e., a factor 100 higher than f_b .

The special case of stable inversion in Fig. 7(a) achieves perfect output tracking, however, the intersample performance is poor as a result of the nonsmooth input signal, see Fig. 6. The special case of multirate inversion in Fig. 7(b) yields perfect state tracking every n = 8 samples, with reasonable intersample performance. The approach in Fig. 7(c) achieves perfect state tracking every $n_1 = 4$ samples and good intersample performance. The proposed approach outperforms the special cases of stable inversion and multirate inversion in terms of the continuous-time error e(t).



Fig. 6. Generated input signals u[k]. The input signal for stable inversion is nonsmooth, whereas the input signals of multirate inversion and the proposed approach are smooth. (a) Stable inversion approach. (b) Multirate inversion approach. (c) Proposed approach.

Second, the performance is evaluated for a variety of solutions. The results are shown in Fig. 8 and quantify Fig. 3. The results show that many of the solutions provided by the approach outperform the special cases of stable inversion and multirate inversion. Note that Fig. 8 only shows results for even numbers n_1 due to the additional lifting step with $\tau = 2$ that is presently used. The solution shown in Figs. 6(c) and 7(c) is the solution that yields the best performance as indicated by • in Fig. 8 and is also used in the experiments in Section V-C. For comparison, the discrete-time norm-optimal solution [6, 4.3] is also shown in Fig. 8. Note that since the boundary effects are negligible, the performance is almost identical to that of the stable inversion approach [6, Sec. 4.2 and 4.3] and, hence, also inferior to that of the proposed approach.

In summary, the simulations show that the intersample performance of the special case of stable inversion is poor and that the performance of the special case of multirate inversion is moderate. Most importantly, the proposed approach offers a variety of options that outperform the two special cases and achieves superior performance. In the next section, the solution that yields the best performance in simulations is validated in experiments.



Fig. 7. Error signals over time. The intersample error signal e(t) (—) and on-sample error signal e[k] (**x**) near t = 0.30 s show that the proposed approach outperforms the other approaches. (a) Stable inversion approach achieves exact output tracking every sample (O). (b) Multirate inversion approach achieves exact state and output tracking every n = 8 samples (O). (c) Proposed approach achieves output tracking every $n_1 = 4$ samples (O).



Fig. 8. Quantification of Fig. 3 through simulation. The number of states n_1 in \underline{H}_1 corresponds to the number of samples between exact on-sample tracking. The approach offers a variety of solutions (**o**). The best choice for the approach $(n_1 = 4, \bullet)$, which is also shown in Figs. 6 and 7, outperforms the stable inversion approach $(n_1 = 0, \bullet)$ and the multirate inversion approach $(n_1 = 8, \bullet)$. For comparison, the discrete-time norm-optimal solution (- -) is also shown and is almost identical to that of the stable inversion solution (**o**).

C. Experimental Validation

In this section, the three solutions shown in Fig. 6 are validated in experiments by application to the motion system shown in Fig. 4.

For the purpose of intersample performance evaluation and due to hardware limitations, the error signal is measured with a



Fig. 9. Experimental results showing the error signal e(t) in the time and frequency domain for stable inversion (—), multirate inversion (·····), the discrete-time norm-optimal solution (– – -), and the proposed approach (– – -). The proposed approach outperforms the special cases of stable inversion and multirate inversion as well as the norm-optimal solution. (a) Time-domain response. (b) Cumulative power spectrum.

sampling frequency of 12.5 kHz, i.e., five times larger than f_b . Besides the feedforward controllers running at the nonequidistant rate, a feedback controller running at an equidistant rate with frequency f_b is used for stabilization. The PID controller with notch filter yields a bandwidth of 10 Hz and a modulus margin of 6 dB. Due to the low bandwidth of the controller, the feedback controller has limited effect on the performance.

The experimental results are shown in Fig. 9. The results show that the proposed approach outperforms the special cases of stable inversion and multirate inversion in terms of the continuoustime error e(t). Note that due to experimental conditions, e.g., model mismatches, there is no exact on-sample tracking every few samples as is the case for the simulations in Section V-B.

In summary, the experiments show the practical applicability of the proposed approach. In particular, the proposed approach outperforms the special cases of stable inversion, multirate inversion as well as norm-optimal control.

VI. CONCLUSION

A discrete-time inversion approach is developed that allows to balance the on-sample and intersample behavior for the purpose of continuous-time performance. The approach is applicable to both LTI and LPTV systems. The multirate inversion and stable inversion approaches are recovered as special cases. Application to an LPTV motion system in both simulations and experiments demonstrates the advantages of the approach.

For LPTV systems, the approach currently involves an additional lifting step, which limits applicability due to constraints on the input and state dimensions, i.e., the state dimension should be an integer multiple of the input dimension. In contrast, stable inversion and multirate inversion can be directly applied to LPTV systems. Future work focuses on an explicit state-space decomposition for LPTV systems to avoid the additional lifting step and thereby potentially increase the performance of the approach.

Obviously, the interest is also in determining the optimal decomposition into H_1 and H_2 a priori in order to avoid exhaustive simulations. Preliminary results show that the best performance is obtained when capturing sampling zeros, introduced by zeroorder-hold discretization, in H_1 , and damped (anti) resonances in H_2 . Ongoing research focuses on more detailed guidelines in finding the optimal decomposition.

APPENDIX A **PROOF OF THEOREM 4**

Due to the modal form of H_{mod} , the A matrix of H_{mod} is block diagonal and the states are decoupled per mode. The matrix T_{per} is a permutation matrix and follows directly from V, V_{\times} and the state ordering of $H_{\rm mod}$. The last n_2 states in (12) are uncontrollable and are redundant since the states are decoupled. Similarly, the first n_1 states in (13) are unobservable and are redundant since the states are decoupled. Hence, H_1 and H_2 are minimal realizations such that $H_{\text{mod}} = H_1 H_2$ in terms of transfer functions. The product H_1H_2 with H_1 and H_2 in (14) is given by

$$H_1 H_2 = \begin{bmatrix} A_1 & B_1 C_2 & B_1 D \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D \end{bmatrix}.$$
 (30)

Using (15)

$$\mathcal{T}(H_1H_2, T_{12}) = \begin{bmatrix} A_1 & -A_1X + B_1C_2 + XA_2 & B_1D + XB_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & -C_1X + C_2 & D \end{bmatrix}$$
(31a)
$$= \begin{bmatrix} A_1 & 0 & B_{2r} \\ 0 & A_2 & B_2 \\ \hline C_1 & C_{1r} & D \end{bmatrix}.$$
(31b)

By definition of T_{per} in (12) and (13), $\mathcal{T}(H_1H_2, T_{\text{per}}^{-1}T_{12}) =$ $\mathcal{T}(\mathcal{T}(H_1H_2, T_{12}), T_{\text{per}}^{-1}) = (A, B, C, D) = H_{\text{mod}}$, which concludes the proof.

APPENDIX B EXAMPLE OF PROCEDURE 7

This appendix demonstrates the steps in Procedure 7 for a simple motion system.

Consider a mass-damper-spring system of which the dynamics are given by

$$m\ddot{q}(t) + d\dot{q}(t) + kq(t) = u(t) \tag{32}$$

with q(t) the displacement, $\dot{q}(t)$ the velocity, $\ddot{q}(t)$ the acceleration, and the parameters mass m = 2 kg, damping constant d = 6 N·s/m, and spring constant k = 4 N/m. The reference trajectory is given by $r(t) = \sin(t)$ and the sample time is $\delta = 1$ ms. For state $x(t) = \begin{bmatrix} \dot{q}(t) \\ q(t) \end{bmatrix}$, i.e., n = 2, the state-space realization in (1) is given by

$$A_c = \begin{bmatrix} -\frac{d}{m} & -\frac{k}{m} \\ 1 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_c = \begin{bmatrix} 0 & 1 \end{bmatrix}. \quad (33)$$

Next, the steps in Procedure 7 are followed.

1) By Lemma 1 follows

$$H_c = C_c (sI - A_c)^{-1} B_c = \frac{B(s)}{A(s)}$$
 (34a)

with

$$B(s) = \frac{\frac{1}{m}}{\frac{1}{m}} = 1 \tag{34b}$$

$$A(s) = \frac{s^2 + \frac{d}{m}s + \frac{k}{m}}{\frac{1}{m}}$$
(34c)

i.e., $b_0 = \frac{1}{m}$, $a_0 = \frac{k}{m}$, and $a_1 = \frac{d}{m}$. Hence, the controllable canonical form in (5) is given by

$$H_{ccf} = \mathcal{T}(H_c, T_{ccf}) \stackrel{s}{=} \begin{bmatrix} 0 & 1 & | & 0 \\ -\frac{d}{m} & -\frac{k}{m} & \frac{1}{m} \\ \hline 1 & 0 & | \end{bmatrix}$$
(34d)

with $T_{ccf}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. 2) Since B(s) = 1, it follows from Theorem 2 that $\hat{x}_{ccf,s}(t) = \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$ and $\hat{x}_{ccf,u}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, hence the desired state in (9) is given by

$$\hat{x}(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}.$$
(35)

Indeed, $y(t) = C_c \hat{x}(t) = \sin(t) = r(t)$ and $\dot{y}(t) = \cos(t) = \dot{r}(t).$

- 3) The desired state of *H* is $\hat{x}[k] = \hat{x}(k\delta)$.
- 4) The modal form in Theorem 4 is obtained by $H_{\rm mod} =$ $\mathcal{T}(H, T_{\text{mod}})$, with

$$T_{\rm mod} = \begin{bmatrix} -2.8284 & -2.8284 \\ -2.2361 & -4.4721 \end{bmatrix}.$$
 (36)

The desired state of $H_{\rm mod}$ is obtained by $\hat{x}_{\rm mod}[k] =$ $T_{\text{mod}}\hat{x}[k].$

5) The state-space decomposition in Theorem 4 requires a nonsingular D, hence a bilinear transformation is used, in particular

$$H_{\text{mod}} = \mathcal{B}(H_{\text{mod}})$$
(37a)
$$\stackrel{s}{=} \begin{bmatrix} 0.001 & 0 & -0.001 \\ 0 & -0.0005 & -0.00079 \\ \hline 0.2502 & -0.3164 & -6.25 \times 10^{-11} \end{bmatrix}.$$

The transfer function decomposition of H_{mod} is obtained through Lemma 3 by selecting the column spaces of the

(37b)

$$\tilde{H}_{1f} \stackrel{s}{=} \begin{bmatrix} 0.001 & 0 & -4 \\ 0 & -0.0005 & 0 \\ \hline 0.2502 & -0.3164 & 1 \end{bmatrix}$$
(38)

$$\tilde{H}_{2f} \stackrel{s}{=} \begin{bmatrix} 0.001 & 0 & -0.001 \\ 0 & -0.0005 & -0.00079 \\ \hline 0 & 0.00016 & -6.25 \times 10^{-11} \end{bmatrix}.$$
 (39)

By selecting $T_{\text{per}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, (12) and (13) are satisfied and the minimal state-space realizations \tilde{H}_1 and \tilde{H}_2 are obtained. Undoing the bilinear transformation yields

$$H_1 = \mathcal{B}^{-1}(\tilde{H}_1) \stackrel{z}{=} \left[\frac{0.998 \left| -5.651 \right|}{0.3536 \left| 0 \right|} \right]$$
(40)

$$H_2 = \mathcal{B}^{-1}(\tilde{H}_2) \stackrel{z}{=} \left[\frac{0.999 - 0.001117}{0.0002235 - 1.25 \times 10^{-7}} \right].$$
(41)

System H_{12} follows from (23).

6) The desired state of H_{12} is given by $\hat{x}_{12}[k] = T_{12}T_{\text{per}}\hat{x}_{\text{mod}}[k]$, with T_{12} in (16) given by

$$T_{12} = \begin{bmatrix} 1 & 1.2649\\ 0 & 1 \end{bmatrix}$$
(42)

with X = 1.2649 the solution to the Sylvester equation (15).

7) The desired state for H_1 is given by $\hat{x}_1[k] = \begin{bmatrix} 1 & 0 \end{bmatrix} \hat{x}_{12}[k]$. This concludes the application of Procedure 7 to the system in (32).

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