Closed form minimum infinity-norm resolution for single-degree kinematically redundant manipulators

Didier Quirin* Valerio Salvucci Moto Kawanobe Travis Baratcart Takafumi Koseki
Department of Electrical Engineering and Information Systems, The University of Tokyo, Japan
Email: {didier, valerio, mot-kwnb,b_travis}@koseki.t.u-tokyo.ac.jp takafumikoseki@ieee.org

***Abstract*—Redundant systems are of interest in engineering because they bring additional capability for a task completion, allowing the system to perform sub-tasks or to improve performances. However, this also means that an extra degree of complexity is added to its resolution. For this purpose, proposed resolution schemes are classically based on the 2-norm minimization, also called pseudo-inverse, whose popularity stems from its easy analytical resolution, but suffers from not considering physical constraints, like input bounds. To tackle this issue, the infinity-norm resolution has been proposed, which determines a minimum-effort solution, taking into consideration individual magnitudes and offering the full physically realisable outputs space. Despite its guaranteed merits, it has only been considered for a few, low-order system because of its lack of analytical resolution. This paper proposes a novel approach on the minimum infinity-norm resolution for single-degree systems, which represent a large part of the most popular redundant configurations. This approach offers a closed-form solution, thus giving an analytical resolution allowing convenient computation and a description of the solution based on parameters of the system. Implementation of this new method is simulated on single-degree kinematically redundant systems to show the superiority of infinity-norm resolution over 2-norm resolution.**

***Index Terms*—redundancy resolution, redundant manipulators, closed form, infinity-norm, minimum effort, kinematics.**

I. Introduction

A system is said to be redundant when it possesses a larger number of degrees of freedom than required to perform a task. We can state the problem of redundancy as follow. The inputs variables $\mathbf{u} \in \mathbb{R}^n$ are translated to the output values $\mathbf{v} \in \mathbb{R}^m$ through the matrix $\mathbf{B} \in \mathbb{R}^{m \times n}$ as:

$$\mathbf{v} = \mathbf{B}\mathbf{u} \tag{1}$$

If $m = n$ and B is non-singular, the solution is unique because B is invertible and bijective. If $n > m$, the m×n system possess more inputs than output, and it is considered redundant. For a redundant system, an infinite number of solutions exist for any particular output, giving additional degrees of freedom to the system. Selecting a useful solution has been a major research topic, which can allow, for example, completion of a sub-task [1], or improve performances [2].

Many practical applications can be classified as single-degree redundant, meaning that if the output is m dimensional, there exist $n = m + 1$ independant inputs to generate this output. In robotics alone, example of applications are biarticular actuation [3], parallel manipulators [4], and kinematically redundant manipulators for applications with 2×3 systems [5], 3×4 systems [6] and 6×7 systems [7].

This paper will deal with this class of single-degree redundant system. Section II of this paper will review existing methods of redundancy resolutions, and in particular previous methods for the minimum infinity-norm resolution and address some of the drawbacks of those resolution methods. Section III introduces the proposed novel approach on the infinity-norm resolution in closed form for single-degree redundant systems. An explicit solution is given as example for a 2×3 kinematically redundant manipulator in section IV. Section V shows simulations of the method for kinematically redundant manipulators and their results. Finally conclusions and outlook are discussed in Section VI.

II. Current redundancy resolution methods

This section summarize the existing redundancy resolution methods and summarize their properties in Table I, with a comparison with the proposed method.

A. Pseudo-inverse based methods and p-norm methods

The most common and easiest method is the resolution using the Moore-Penrose pseudo-inverse, also called 2-norm resolution. It resolves (1) as:

$$\mathbf{u} = \mathbf{B}^{\dagger}\mathbf{v} \tag{2}$$

Where, if $\mathbf{B}$ is full row rank, the pseudo inverse $\mathbf{B}^{\dagger}$ of $\mathbf{B}$ is defined by:

$$\mathbf{B}^{\dagger} = \mathbf{B}^T(\mathbf{B}\mathbf{B}^T)^{-1} \tag{3}$$

The pseudo-inverse gives a unique continuous solution for $\mathbf{u}$ which correspond to the 2-norm optimization of the input subject to (1):

$$\begin{array}{ll} \min & \sqrt{u_1^2 + u_2^2 + ... + u_n^2} \\ \text{s.t.} & \mathbf{v} = \mathbf{B}\mathbf{u} \end{array} \tag{4}$$

This optimization process can be identified physically as the minimization of energy in the system, since it minimizes the sum of squares. However, the pseudo-inverse is far more used for its analytical properties than for its physical properties. A physical system often presents bounds and physical conditions, such as bounds on some input contributions. Since the pseudo-inverse only minimize an average of inputs, it does not take into consideration the individual contribution of inputs.

An extensive research has been put on extending the pseudo-inverse to integrate those physical conditions. A notable example is the weighted pseudo-inverse [5], where a criteria matrix $\mathbf{W}$ puts priority and bounds on variables, as:

$$\mathbf{B}^{\dagger} = \mathbf{W}^{-1}\mathbf{B}^{T}(\mathbf{B}\mathbf{W}^{-1}\mathbf{B}^{T})^{-1} \tag{5}$$

Other examples include Least-squares with clipping [8], Redistributed Pseudo-Inverse [9] and more recently Cascaded Generalized Inverse [10]. However all those methods suffer the same problem, which is that they cannot offer full output utilization, meaning that, for a given output, the resolved inputs with those methods are not always in allowable ranges, although a suitable solution exists.

In litterature for redundant systems, there has been little attention dedicated to using norms other than the 2-norm. In [11], the usage of p-norms was investigated to address definition of subtasks after considering the redundancy resolution with pseudo-inverse. p-norms are defined as:

$$||u||_p = \left(\sum_{i=1}^{n} |u_i|^p\right)^{\frac{1}{p}} \tag{6}$$

Direct resolution of (1) with a p-norm has only been investigated in the case of the infinity-norm, which corresponds to p approaching infinity. Although high-index norms can give a fair approximation of this infinity-norm, their usage has not been developed because they lack physical meaning, in comparison with 2-norm and infinity-norm, and require optimization solvers to find a solution, and thus lack analytical resolution.

B. Infinity-norm resolution methods

Infinity-norm resolution has been investigated for several decades. It was first introduced under the Minimum-Effort problem by J.A Cadzow [12]. In physical systems, physical parameters have to be optimized, such as joint velocities in the case of kinematic redundancy. As those joint are actuated, it is reasonable to consider that their individual contribution has to be limited. We cannot allow an actuator to exceed its speed bound, or one joint to support a large share of the total output. Whereas the pseudo-inverse minimize an average of inputs, without taking into consideration those limits, the infinity-norm resolution relates to the magnitude of individual variables, and thus can distribute contribution between inputs effectively. The property of infinity-norm also allows this resolution to attain the full physically realisable output, meaning that, if a component of a solution for a specific task given by this resolution exceeds its physical limit, the desired task is physically unfeasible. Following J.A Cadzow, who proposed an algorithm converging to the optimal infinity-norm solution, I. C. Shim and Y. S. Yoon [13] offered an equivalent algorithm, but with a geometrical approach. Those algorithms does not insure convergence to the optimal solution. An algorithm was proposed by I. Ha and J. Lee [14], using the equal magnitude property (See III), which ensures convergence of the algorithm. However this method requires computation of the minimum 2-norm to find the optimal infinity-norm solution. Another notable work by Y. Zhang [15] proposes to compute the optimal infinity-norm solution using quadratic programming. Those methods lack analytical resolution compared to the pseudoinverse which gives a closed form solution. The work of I. A. Gravagne and I. D. Walker [16] proposed the construction of an infinity inverse, giving a closed-form solution, on the condition that an algorithm (such as one of the previously mentioned work) gives the maximum value of the optimal infinity-form beforehand. Its utility is merely for analysing the infinity-norm problem. Recently, a closed-form solution was given to treat biarticular actuation in [17]. This is a specific 2×3 system, where the matrix B in (1) has fixed values. An attempt to give a closed form solution for a general 2×3 system was made in [18], but it did not propose a mathematical proof and in fact does not give all possible cases of closed form solution.

III. Novel minimum infinity-norm resolution

We propose a novel method to finding a closed form solution for single-degree redundant systems.

First let us restate the infinity norm resolution problem and explain the reason for using this norm. The infinity norm resolution corresponds to solving the following problem:

$$\begin{aligned} &\min ||u||_\infty \\ \text{s.t.} \quad &\mathbf{v}{=}\mathbf{B}\mathbf{u} \end{aligned} \tag{7}$$

In our case, for a single-degree redundant system, we have the input values $\mathbf{u} \in \mathbb{R}^{m+1}$, the task space values $\mathbf{v} \in \mathbb{R}^{m}$ and the matrix $\mathbf{B} \in \mathbb{R}^{m\times(m+1)}$ is full rank m. The optimization problem min $||\mathbf{u}||_\infty$ can be rewritten for better understanding as:

$$\min \quad [\max(|u_1|, |u_2|, ..., |u_{m+1}|)] \tag{8}$$

Where $\mathbf{u} = (u_1, u_2, ..., u_{m+1})^T$.

A graphical comparison between infinity norm and 2 norm optimization in 2-dimension is given in Fig. 1. The dashed line represent the solution space of the linear system $\mathbf{v} = \mathbf{B}\begin{pmatrix} x \\ y \end{pmatrix}$ in the 2-dimensional case. The circle and square represent respectively 2-norm and infinity-norm. Indeed, if we consider representing the norms graphically, we can define an equinorm surface $\mathcal{B}(\mathrm{r}, \mathrm{p})$ such as:

$$\mathcal{B}(\mathrm{r}, \mathrm{p}) = \{||z||_p = r,\ \mathbf{z} \in \mathbb{R}^n\} \tag{9}$$

where $r > 0$ is the equinorm radius. For the 2-norm and infinity-norm, their respective graphical representation $\mathcal{B}(\mathrm{r}, 2)$ and $\mathcal{B}(\mathrm{r}, \infty)$ are a circle and square in 2-dimension (in 3-dimension it would be a sphere and a cube). In Fig. 1(a), we can observe that, for a given solution space, the maximum input, here is lower for the infinity-norm solution (here $x_\infty = y_\infty$) than for the 2-norm solution (here x_2). This emphasize the property:

$$\max\{|x_\infty|, |y_\infty|\} \leq \max\{|x_2|, |y_2|\} \tag{10}$$

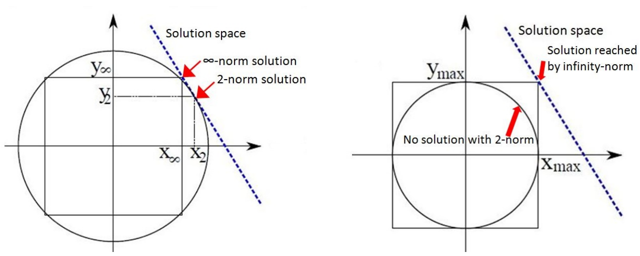
Table I
COMPARISON OF METHODS

	Pseudo inverse	Weighted pseudo inverse	Least square with clipping Redistributed Pseudo inverse	CGI	P-norm	Infinity-norm	Proposed method
Continuity	Yes	Yes	Yes	No	Yes	No	No
Uniqueness	Yes	Yes	Yes	Yes	Yes	No	No
Closed form solution	Yes	Yes	Yes	Yes	No	No	Yes
Individual contribution of inputs	No	Yes	No	No	No	Yes	Yes
Full reachable output space	No	No	No	No	Close to full	Full	Full
Computation	Fast	Fast	Fast	Average	Slow	Slow	Fast

When the system has an allowable range for inputs, here x and y, defined as:

$$
\begin{aligned}
-x_{\max} \leq x \leq x_{\max} \\
-y_{\max} \leq y \leq y_{\max}
\end{aligned}
\tag{11}
$$

we can see from Fig. 1(b), that infinity-norm resolution reaches a greater solution space, as the circle can not be expanded to reach the solution space due to the bounds $x_{\max}$ and $y_{\max}$.



(a) (b)

Figure 1. Graphical comparison between infinity-norm resolution and 2-norm resolution for a 2 dimensional problem

For our case of single-degree redundant system, a property of minimum infinity-norm will be fundamental to find a closed form solution. This property was discovered in the works [14] and [16], and is called Equal Magnitude Property: an optimal minimum infinity norm solution always exists with at least $n-m+1$ components of equal magnitude and this magnitude is the maximum magnitude of all its components.

By applying this property to a single-degree $m \times (m+1)$ redundant system, we can state: Let $\mathbf{u}^* = (u_1^*, u_2^*, ..., u_{m+1}^*)^T \in \mathbb{R}^{m+1}$ be an optimal minimum infinity norm solution. Then at least 2 components of $\mathbf{u}^*$ will be equal to $\pm||\mathbf{u}^*||_\infty$ and the other components have a lower or equal magnitude. We can state this as:

$$\exists (i,j) \in [1, m+1] \, , \, |u_i^*| = |u_j^*| \tag{12}$$

$$\forall k \neq i,j \, , \, |u_k^*| \leq |u_i^*| = |u_j^*| \tag{13}$$

The first equation of the Equal Magnitude Property, (12), allows us to eliminate the extra degree of freedom of the redundant system and select up to $m \times (m+1)$ solution candidates, among which one is the optimal infinity-norm solution. Although this condition can imply that more than 2 components have equal magnitude, knowing that at least 2 are equal is enough for us to find all the possible solutions. This is done by rewritting (1) into $m \times (m+1)$ candidate linear systems, given as:

$$
\begin{bmatrix} \mathbf{v} \\ 0 \end{bmatrix} = \mathbf{B}'_{i,j+} \mathbf{u} = \begin{bmatrix} \mathbf{B} \\ \delta_{i,j+} \end{bmatrix} u \quad \text{or} \quad \begin{bmatrix} \mathbf{v} \\ 0 \end{bmatrix} = \mathbf{B}'_{i,j-} \mathbf{u} = \begin{bmatrix} \mathbf{B} \\ \delta_{i,j-} \end{bmatrix} \mathbf{u}
\tag{14}
$$

Where $\delta_{i,j+} = (\delta_{1+}, \delta_{2+}, ..., \delta_{m+1+})^T$ and $\delta_{i,j-} = (\delta_{1-}, \delta_{2-}, ..., \delta_{m+1-})^T$ with:

$$
\delta_{k+} = \begin{cases} 0 & \text{if } k \neq i,j \\ 1 & \text{if } k = i,j \end{cases} \quad \text{or} \quad \delta_{k-} = \begin{cases} 0 & \text{if } k \neq i,j \\ 1 & \text{if } k = i \\ -1 & \text{if } k = j \end{cases}
\tag{15}
$$

Let us call $|\mathbf{B}_k|$ a $m \times m$ minor of matrix $\mathbf{B}$ created by removing column k. The existence of a solution is insured by the fact that $\mathbf{B}$ is full rank, meaning that at least one of its minor $|\mathbf{B}_k|$ is non-zero. This means that at least one $\mathbf{B}'_{i,j}$ is a square $m+1$ full rank matrix, and is consequently invertible. Let us introduce $\mathbf{B}_{v,i}$ the matrix formed by replacing the i-th column of $\mathbf{B}$ by the column vector $\mathbf{v}$. We define $|\mathbf{B}_{v,i,k}|$ a $m \times m$ minor of matrix $\mathbf{B}_{v,i}$ created by removing column k. For $(i,j) \in [1, m+1]$, such as $1 \leq i \leq j \leq m+1$, we can set all possible $m \times (m+1)$ candidate $\mathbf{u}_{i,j}^* = (u_{i,j,1}^*, u_{i,j,2}^*, ..., u_{i,j,m+1}^*)^T$ solutions as:

If $u_{i,j,i}^* = u_{i,j,j}^*$

$$
\begin{cases}
u_{i,j,k}^* = \frac{(-1)^i |\mathbf{B}_{v,k,i}| + (-1)^j |\mathbf{B}_{v,k,j}|}{(-1)^i |\mathbf{B}_i| + (-1)^j |\mathbf{B}_j|} & k \neq i,j \\
u_{i,j,k}^* = \frac{(-1)^i |\mathbf{B}_{v,j,i}|}{(-1)^i |\mathbf{B}_i| + (-1)^j |\mathbf{B}_j|} & k = i,j
\end{cases}
\tag{16}
$$

If $u_{i,j,i}^* = -u_{i,j,j}^*$

$$
\begin{cases}
u_{i,j,k}^* = \frac{(-1)^i |\mathbf{B}_{v,k,i}| - (-1)^j |\mathbf{B}_{v,k,j}|}{(-1)^i |\mathbf{B}_i| - (-1)^j |\mathbf{B}_j|} & k \neq i,j \\
u_{i,j,k}^* = \frac{(-1)^i |\mathbf{B}_{v,k,i}|}{(-1)^i |\mathbf{B}_i| + (-1)^j |\mathbf{B}_j|} & k = i,j
\end{cases}
\tag{17}
$$

Now that we have all possible candidate solutions $u_{i,j}^*$, among which one is the optimal infinity-norm solution, we can eliminate incorrect ones. From (16) and (17), we can see that some possible solutions $u_{i,j}^*$ can be discarded, when either $(-1)^i |\mathbf{B}_i| + (-1)^j |\mathbf{B}_j| = 0$ or $(-1)^i |\mathbf{B}_i| - (-1)^j |\mathbf{B}_j| = 0$. This condition is not sufficient to reach the single optimal solution, but can be useful in a computational algorithm to discard unfeasible solutions right away and have less to process. We use (13) to reduce the number of candidates by checking that for each $\mathbf{u}_{i,j}^*$ candidate solution:

$$|u_{i,j,k}^*| \leq |u_{i,j,i}^*| = |u_{i,j,j}^*| \quad \text{for } k \neq i,j \tag{18}$$

Which can be rewritten using (16) and (17):

If $u_{i,j,i}^* = u_{i,j,j}^*$

$$\left|(-1)^i|\mathbf{B}_{v,k,i}| + (-1)^j|\mathbf{B}_{v,k,j}|\right| \leq |\mathbf{B}_{v,k,i}| \tag{19}$$

If $u_{i,j,i}^* = u_{i,j,j}^*$

$$\left|(-1)^i|\mathbf{B}_{v,k,i}| - (-1)^j|\mathbf{B}_{v,k,j}|\right| \leq |\mathbf{B}_{v,k,i}| \tag{20}$$

This step provides that the remaining candidate solutions have their equal magnitude components equal to the maximum magnitude of the optimal infinity-norm solution.

The last step of selection compares the components of the remaining candidate solutions. $\mathbf{u}_{i,j}^*$ will be selected as the optimal solution if it verifies:

$$\forall k \neq i,j,\ |u_{i,j,k}^*| \leq |u_{l,m,k}^*| \text{ for } (i,j) \neq (l,m) \tag{21}$$

The optimal infinity-norm closed form solution $\mathbf{u}_{i,j}^{*,\text{opt}}$ will have its components $u_{i,j,k}^{*,\text{opt}}$ given using (16) or (17).

It is to be noted that (19) or (20) combined with (21), can be rewritten using only minors $|\mathbf{B}_k|$ of matrix $\mathbf{B}$ and minors $|\mathbf{B}_{v,i,k}|$ of matrices $\mathbf{B}_{v,i}$ formed by replacing the i-th column of B by the column vector v. This property can be exploited to give an elegant solution in lower dimension systems, as will be shown in the next section for a 2×3 system.

IV. 2X3 Kinematically Redundant Manipulator Explicit Closed Form Solution

For kinematic redundancy, the problem (1) is formulated as follow. Here the inputs are the joint velocities, $\dot{\mathbf{q}}$, and the output is the end-effector velocity, $\mathbf{v}_{\text{eff}}$. They relate through the Jacobian matrix, $\mathbf{J}$ as:

$$\mathbf{v}_{\text{eff}} = \mathbf{J}\dot{\mathbf{q}} \tag{22}$$

Where:

$$\mathbf{J}(i,j) = \frac{\delta \mathbf{x}_{\text{eff},i}}{\delta \mathbf{q}_j} \tag{23}$$

With $\mathbf{x}_{\text{eff}}$ the end-effector position and $\mathbf{q}$ the joint angles. In the case of the 2×3 kinematically redundant manipulator, there are three joint space variables to actuate the 2-dimensional end-effector velocity, meaning that $\mathbf{v}_{\text{eff}} = (\dot{\mathbf{x}}, \dot{\mathbf{y}})^T$ and $\dot{\mathbf{q}} = (\dot{\mathbf{q_1}}, \dot{\mathbf{q_2}}, \dot{\mathbf{q_3}})^T$.

Following section III, we set $|\mathbf{J}_1|, |\mathbf{J}_2|, |\mathbf{J}_3|$ the respective minors of matrix $\mathbf{J}$, and $|\mathbf{J}_{1v}|, |\mathbf{J}_{2v}|, |\mathbf{J}_{3v}|$ the minors formed by taking the determinant of respectively column 1,2,3 of $\mathbf{J}$ with vector $\mathbf{v}_{\text{eff}}$. The closed form solution $\dot{\mathbf{q}}^* = (\dot{\mathbf{q_1}}^*, \dot{\mathbf{q_2}}^*, \dot{\mathbf{q_3}}^*)^T$ is given:

$$\dot{\mathbf{q_1}}^* = \begin{cases} \frac{-|\mathbf{J}_{3v}|}{|\mathbf{J}_2|+|\mathbf{J}_1|} & \text{case 1} \\ \frac{-|\mathbf{J}_{3v}|}{|\mathbf{J}_2|-|\mathbf{J}_1|} & \text{case 2} \\ \frac{-|\mathbf{J}_{2v}|}{|\mathbf{J}_3|-|\mathbf{J}_1|} & \text{case 3} \\ \frac{-|\mathbf{J}_{2v}|}{|\mathbf{J}_3|+|\mathbf{J}_1|} & \text{case 4} \\ \frac{-|\mathbf{J}_{3v}|-|\mathbf{J}_{2v}|}{|\mathbf{J}_3|+|\mathbf{J}_2|} & \text{case 5} \\ \frac{|\mathbf{J}_{3v}|-|\mathbf{J}_{2v}|}{|\mathbf{J}_3|-|\mathbf{J}_2|} & \text{case 6} \end{cases} \tag{24}$$

$$\dot{\mathbf{q_2}}^* = \begin{cases} \frac{-|\mathbf{J}_{3v}|}{|\mathbf{J}_2|+|\mathbf{J}_1|} & \text{case 1} \\ \frac{|\mathbf{J}_{3v}|}{|\mathbf{J}_2|-|\mathbf{J}_1|} & \text{case 2} \\ \frac{|\mathbf{J}_{3v}|+|\mathbf{J}_{1v}|}{|\mathbf{J}_3|-|\mathbf{J}_1|} & \text{case 3} \\ \frac{|\mathbf{J}_{3v}|-|\mathbf{J}_{1v}|}{|\mathbf{J}_3|+|\mathbf{J}_1|} & \text{case 4} \\ \frac{|\mathbf{J}_{1v}|}{|\mathbf{J}_3|+|\mathbf{J}_2|} & \text{case 5} \\ \frac{|\mathbf{J}_{1v}|}{|\mathbf{J}_3|-|\mathbf{J}_2|} & \text{case 6} \end{cases} \tag{25}$$

$$\dot{\mathbf{q_3}}^* = \begin{cases} \frac{-|\mathbf{J}_{2v}|-|\mathbf{J}_{1v}|}{|\mathbf{J}_2|+|\mathbf{J}_1|} & \text{case 1} \\ \frac{|\mathbf{J}_{2v}|-|\mathbf{J}_{1v}|}{|\mathbf{J}_2|-|\mathbf{J}_1|} & \text{case 2} \\ \frac{-|\mathbf{J}_{2v}|}{|\mathbf{J}_3|-|\mathbf{J}_1|} & \text{case 3} \\ \frac{|\mathbf{J}_{2v}|}{|\mathbf{J}_3|+|\mathbf{J}_1|} & \text{case 4} \\ \frac{|\mathbf{J}_{1v}|}{|\mathbf{J}_3|+|\mathbf{J}_2|} & \text{case 5} \\ \frac{-|\mathbf{J}_{1v}|}{|\mathbf{J}_3|-|\mathbf{J}_2|} & \text{case 6} \end{cases} \tag{26}$$

Where the 6 cases are defined:

$$\text{case 1} := \quad \text{abs}|\mathbf{J}_{3v}| \geq \text{abs}(|\mathbf{J}_{1v}| + |\mathbf{J}_{2v}|) \text{ and } (|\mathbf{J}_1| \geq 0, |\mathbf{J}_2| \geq 0 \text{ or } |\mathbf{J}_1| \leq 0, |\mathbf{J}_2| \leq 0))$$

$$\text{case 2} := \quad \text{abs}|\mathbf{J}_{3v}| \geq \text{abs}(|\mathbf{J}_{1v}| - |\mathbf{J}_{2v}|) \text{ and } (|\mathbf{J}_1| \leq 0, |\mathbf{J}_2| \geq 0 \text{ or } |\mathbf{J}_1| \geq 0, |\mathbf{J}_2| \leq 0))$$

$$\text{case 3} := \quad \text{abs}|\mathbf{J}_{2v}| \geq \text{abs}(|\mathbf{J}_{1v}| + |\mathbf{J}_{3v}|) \text{ and } (|\mathbf{J}_1| \geq 0, |\mathbf{J}_3| \leq 0 \text{ or } |\mathbf{J}_1| \leq 0, |\mathbf{J}_2| \geq 0))$$

$$\text{case 4} := \quad \text{abs}|\mathbf{J}_{2v}| \geq \text{abs}(|\mathbf{J}_{1v}| - |\mathbf{J}_{3v}|) \text{ and } (|\mathbf{J}_1| \geq 0, |\mathbf{J}_3| \geq 0 \text{ or } |\mathbf{J}_1| \leq 0, |\mathbf{J}_3| \leq 0))$$

$$\text{case 5} := \quad \text{abs}|\mathbf{J}_{1v}| \geq \text{abs}(|\mathbf{J}_{2v}| + |\mathbf{J}_{3v}|) \text{ and } (|\mathbf{J}_2| \geq 0, |\mathbf{J}_3| \geq 0 \text{ or } |\mathbf{J}_2| \leq 0, |\mathbf{J}_3| \leq 0))$$

$$\text{case 6} := \quad \text{abs}|\mathbf{J}_{1v}| \geq \text{abs}(|\mathbf{J}_{2v}| - |\mathbf{J}_{3v}|) \text{ and } (|\mathbf{J}_2| \geq 0, |\mathbf{J}_3| \leq 0 \text{ or } |\mathbf{J}_2| \leq 0, |\mathbf{J}_3| \geq 0))$$

Proof of closed-form solution for this 2×3 configuration is given in appendix.

V. Simulations and Results

The proposed method is implemented to resolve kinematic redundancy in 2×3 and 3×4 configurations for robotic manipulators. In the case of the 2×3 configuration, which uses the closed form solution given in IV, the computational advantage of the proposed method is given.

A. 2×3 *configuration for kinematic redundancy implementation*

For this simulation, we use the closed-form solution developed in IV. We set the initial configuration of a 3-link manipulator with unit-length links with the joint positions $\mathbf{q} = [\frac{\pi}{32}, \frac{\pi}{4}, \frac{\pi}{4}]$. Fig. 2 shows the 3-link manipulator. We command a movement in the direction of negative x with a velocity magnitude of 2 units/sec from initial position. We set the joints velocities limit to 1 rad/sec. Fig. 3 shows the resolution of the three joint velocities for both 2-norm resolution and infinity-norm resolution. The simulation shows that until $t \approx 0.82s$, both 2-norm resolution and infinity-norm resolution produces a feasible trajectory. At $t \approx 0.82s$, 2-norm resolution fails to produce realizable joint velocities, because the velocity $\dot{\mathbf{q_1}}$ reaches the limit. However infinity-norm resolution continues to produce a realizable trajectory.

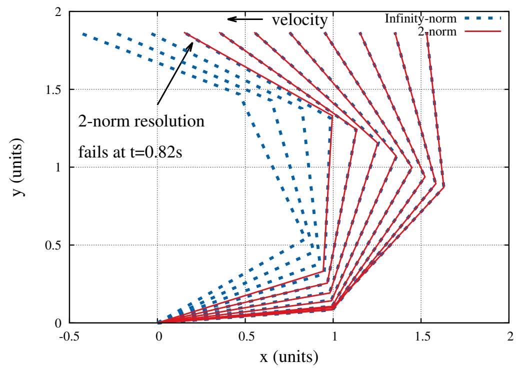


Figure 2. Trajectory of a 3-link planar arm implementing the proposed infinity-norm optimization

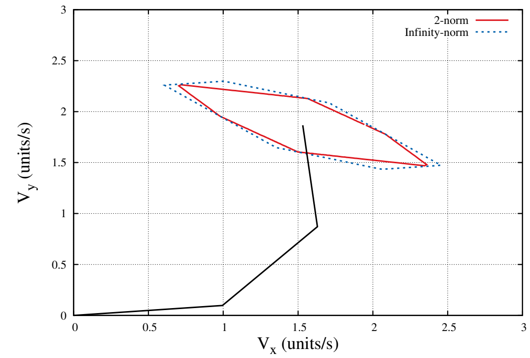


Figure 4. Maximum achievable joint velocities using 2-norm and infinity-norm resolution

It is to be noted when observing Fig. 2 that both trajectories are almost similar until the contribution of the first joint becomes large. Since infinity-norm resolution reallocates the contribution to other joints, the trajectory of the 3-link planar arm using infinity-norm resolution starts to differ from 2-norm resolution.

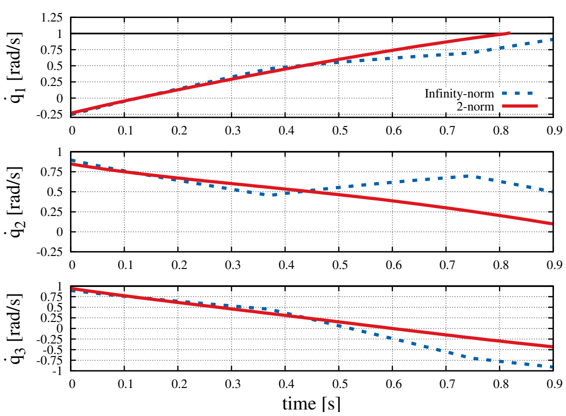


Figure 3. Resolved joint velocities for 3-link planar arm

Fig. 4 plots the maximum achievable velocities using 2-norm and infinity-norm resolution, with a scaling of $\frac{1}{5}$ for the same manipulator in configuration $\mathbf{q} = [\frac{\pi}{32}, \frac{\pi}{4}, \frac{\pi}{4}]$. Following the theoretical analysis of infinity-norm in section III, this figure confirms that the maximum velocity attainable with infinity-norm resolution is always greater than to that when using 2-norm resolution. The points at which these two methods are equal are when 2-norm resolution reaches the joints limits simultaneously. The computational efficiency of the closed form resolution method given in IV has been calculated by running the method for random $\mathbf{v}_{\text{eff}}$ vectors and random $\mathbf{J}$ matrix 1000 times and compared to a traditional algorithm which computes all possible solution candidates. It is found that the proposed method is at least 35% faster than a traditional algorithm.

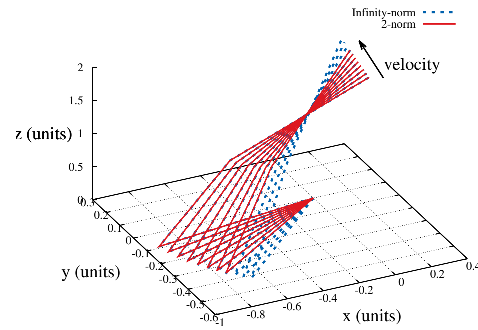


Figure 5. Trajectory of a 4-link arm implementing the proposed infinity-norm optimization

B. 3×4 configuration for kinematic redundancy implementation

The closed form solution for this configuration is not shown, as it is a direct derivation from III in the same manner as IV. For the 3×4 configuration, the derivation will give 12 cases for $\dot{\mathbf{q}}^* = (\dot{\mathbf{q}}_1^*, \dot{\mathbf{q}}_2^*, \dot{\mathbf{q}}_3^*, \dot{\mathbf{q}}_4^*)^T$. Here, we choose an anthropomorphic arm, which consists in a 3-link manipulator presented in section V-A rotated about the z axis. Here we set the initial configuration of the anthropomorphic arm with unit-length links and starting joint positions $\mathbf{q} = [0.55\pi, 0.05\pi, 0.60\pi, 0.20\pi]$, as shown in Fig. 5. The arm is commanded to move with end effector velocity $[0, 1, 1]$ units/sec from initial position for 0.2 seconds. The joints velocity limits are set to 1 rad/sec, except for the rotating angle about the z axis since it uniquely defines one degree of freedom. The simulation shows that at $t \approx 0.15s$, 2-norm resolution fails to produce realizable joint velocities, but infinity-norm resolution continues to produce a realizable trajectory.

VI. Conclusions and Outlook

Resolution of redundant systems often uses the pseudo-inverse method, also called 2-norm, because of its analytical resolution, ensuring a closed form solution when the system has full rank. However this approach does not take into consideration contribution of individual inputs and fails to reach the full physically realisable output space. The infinity-norm resolution approach was introduced to answer those issues, but the lack of closed form solution has so far rendered this method unpopular.

This paper has introduced a novel approach on the minimum infinity-norm resolution for single-degree systems (m outputs and $m+1$ inputs). Compared to previous research on infinity-norm resolution, this new method proposes selective conditions to reach a closed-form optimal solution. Those conditions allow a simple description of the optimal solution based on output and system values and can be exploited directly in control schemes. Also, since it does not use a recursive or complicated algorithm, as proposed in previous papers, this method allows for ease of computation, which was calculated to be at least 35% faster in the $\mathbf{2} \times \mathbf{3}$ configuration compared with a traditional algorithm comparing all possible candidates.

Demonstration of implementation and utility of this method has been performed by considering kinematically redundant robotic manipulators. Simulations have shown that the maximum achievable velocities using infinity-norm resolution is always greater than or equal to that when using 2-norm resolution. Also, when joints velocity limits have to be considered, the infinity-norm resolution is able to produce further realizable trajectories.

Future works include experimental implementation and verification of the proposed method. Although a large portion of redudant systems are single-degree, an attempt to generalize the method to redundant system with greater degree-of-freedom is also a clear point of future work.

Appendix

The demonstration on how to determine the closed form solution for the 2×3 configuration is given here.

Proof: Let $\mathbf{J}$ be a 2×3 full row rank matrix which translates input variables $\dot{\mathbf{q}} = (\dot{\mathbf{q_1}}, \dot{\mathbf{q_2}}, \dot{\mathbf{q_3}})^T$ to task space values $\mathbf{v}_{\text{eff}} = (\dot{\mathbf{x}}, \dot{\mathbf{y}})^T$. We define $|\mathbf{J}_1|, |\mathbf{J}_2|, |\mathbf{J}_3|$ the respective minor of matrix $\mathbf{J}$ and $|\mathbf{J}_{1v}|, |\mathbf{J}_{2v}|, |\mathbf{J}_{3v}|$ the minors formed by taking the determinant of respectively column 1,2,3 of $\mathbf{J}$ with vector $\mathbf{v}_{\text{eff}}$ following III. As defined in IV, in the 2×3 configuration, we have 6 possible candidates:

$$\begin{cases} \dot{\mathbf{q}}_{\mathbf{a}}^* & \text{if} \quad \dot{\mathbf{q_1}} = \dot{\mathbf{q_2}} \\ \dot{\mathbf{q}}_{\mathbf{b}}^* & \text{if} \quad \dot{\mathbf{q_1}} = -\dot{\mathbf{q_2}} \\ \dot{\mathbf{q}}_{\mathbf{c}}^* & \text{if} \quad \dot{\mathbf{q_1}} = \dot{\mathbf{q_3}} \\ \dot{\mathbf{q}}_{\mathbf{d}}^* & \text{if} \quad \dot{\mathbf{q_1}} = -\dot{\mathbf{q_3}} \\ \dot{\mathbf{q}}_{\mathbf{e}}^* & \text{if} \quad \dot{\mathbf{q_2}} = \dot{\mathbf{q_3}} \\ \dot{\mathbf{q}}_{\mathbf{f}}^* & \text{if} \quad \dot{\mathbf{q_2}} = \dot{\mathbf{q_3}} \end{cases} \tag{27}$$

Following (19), the solution $\dot{\mathbf{q}}_{\mathbf{a}}^*$ has components:

$$\dot{\mathbf{q}}_{\mathbf{a}}^* = \begin{pmatrix} \dot{\mathbf{q}}_{\mathbf{a,1}}^* = \frac{-|\mathbf{J}_{3v}|}{|\mathbf{J}_2|+|\mathbf{J}_1|} \\ \dot{\mathbf{q}}_{\mathbf{a,2}}^* = \frac{-|\mathbf{J}_{3v}|}{|\mathbf{J}_2|+|\mathbf{J}_1|} \\ \dot{\mathbf{q}}_{\mathbf{a,3}}^* = \frac{-|\mathbf{J}_{2v}|-|\mathbf{J}_{1v}|}{|\mathbf{J}_2|+|\mathbf{J}_1|} \end{pmatrix} \tag{28}$$

For this solution to be the infinity-norm solution, first we need $|\mathbf{J}_1| + |\mathbf{J}_2| \neq 0$, and it needs to follow (18), meaning: $|\dot{\mathbf{q}}_{\mathbf{a,3}}^*| \leq |\dot{\mathbf{q}}_{\mathbf{a,1}}^*| = \dot{\mathbf{q}}_{\mathbf{a,2}}^*|$. This can be rewritten as: $\text{abs}|\mathbf{J}_{3v}| \geq \text{abs}(|\mathbf{J}_{1v}| + |\mathbf{J}_{2v}|)$. Another condition is also given by (21), giving 5 inequations:

$$\begin{aligned} \text{abs}\Big(\frac{-|\mathbf{J}_{3v}|}{|\mathbf{J}_2|+|\mathbf{J}_1|}\Big) &\leq \text{abs}\Big(\frac{-|\mathbf{J}_{3v}|}{|\mathbf{J}_2|-|\mathbf{J}_1|}\Big) \\ \text{abs}\Big(\frac{-|\mathbf{J}_{3v}|}{|\mathbf{J}_2|+|\mathbf{J}_1|}\Big) &\leq \text{abs}\Big(\frac{-|\mathbf{J}_{2v}|}{|\mathbf{J}_3|-|\mathbf{J}_1|}\Big) \\ \text{abs}\Big(\frac{-|\mathbf{J}_{3v}|}{|\mathbf{J}_2|+|\mathbf{J}_1|}\Big) &\leq \text{abs}\Big(\frac{-|\mathbf{J}_{2v}|}{|\mathbf{J}_3|+|\mathbf{J}_1|}\Big) \\ \text{abs}\Big(\frac{-|\mathbf{J}_{3v}|}{|\mathbf{J}_2|+|\mathbf{J}_1|}\Big) &\leq \text{abs}\Big(\frac{-|\mathbf{J}_{1v}|}{|\mathbf{J}_3|+|\mathbf{J}_2|}\Big) \\ \text{abs}\Big(\frac{-|\mathbf{J}_{3v}|}{|\mathbf{J}_2|+|\mathbf{J}_1|}\Big) &\leq \text{abs}\Big(\frac{-|\mathbf{J}_{1v}|}{|\mathbf{J}_3|-|\mathbf{J}_2|}\Big) \end{aligned} \tag{29}$$

This set of inequations can be simplified as ($|\mathbf{J}_1| \geq 0, |\mathbf{J}_2| \geq 0$ or $|\mathbf{J}_1| \leq 0, |\mathbf{J}_2| \leq 0$)). Following those two conditions, we have: case 1 $:= \text{abs}|\mathbf{J}_{3v}| \geq \text{abs}(|\mathbf{J}_{1v}| + |\mathbf{J}_{2v}|)$ and ($|\mathbf{J}_1| \geq 0, |\mathbf{J}_2| \geq 0$ or $|\mathbf{J}_1| \leq 0, |\mathbf{J}_2| \leq 0$)).

A similar method shows the proof for the remaining 5 cases. ■

References

[1] A. Maciejewski, "Obstacle avoidance for kinematically redundant manipulators in dynamically varying environments," in The Int. Journal of Robotics Research, vol. 4, no. 3, pp. 109-117, 1985.

[2] S. Chiaverini, "Singularity-robust task-priority redundancy resolution for real-time kinematic control of robot manipulators," in IEEE Trans. Robot. Autom., vol. 13, no. 3, pp. 398-410, 1997.

[3] V. Salvucci, Y. Kimura, O. Sehoon, and Y. Hori, "BiWi: Bi-articularly actuated and wire driven robot arm," in Proc. IEEE Int. Conf. Mechatronics, pp. 827-832, 2011.

[4] J. Arata, N. Ikedo, and H. Fujimoto, "Force producibility improvement of redundant parallel mechanisms for haptic applications," IEEE/RSJ Int. Conf. on Intelligent Robots and Systems, pp. 2145-2150, 2011.

[5] J. Hollerbach, and K.C. Suh, "Redundancy resolution of manipulators through torque optimization," IEEE Journal of Robotics and Automation, vol. RA-3, no. 4, pp. 308-316, 1987.

[6] T. Yoshikawa, "Manipulability of robotics mechanisms," in Int. Journal of Robotics Research, vol. 4(2), pp. 303-309, 1985.

[7] G. Hirzinger, N. Sporer, A. Albu-Schaffer, M. Hahnle, R. Krenn, A. Pascucci, and M. Schedl, "DLR's torque-controlled light weight robot III - are we reaching the technological limits now?," in IEEE Int. Conf. on Robotics and Automation, pp 1710-1716, 2002.

[8] J.A.M Pertersen and M. Bodson," Constrained quadratic programming techniques for control allocation," IEEE Trans. Control Syst. Technol., vol. 14, no 1, pp. 91-98, 2006.

[9] J. C. Virnig and D. S. Bodden, "Multivariable control allocation and control law conditioning when control effectors limit (STVOL aircraft)," in AIAA Guidance, Navigation, and Control Conf., pp. 572-582, 1994.

[10] T. Baratcart, V. Salvucci, and T. Koseki, "Dynamic analysis of continuous cascaded generalized inverse resolution of kinematically redundant manipulators with flexible joints," in The 13th European Control Conf., Strasbourg, France, 2014.

[11] C. A. Klein, C-H. Huang, "Review of pseudo-inverse control for use with kinematically redundant manipulators," in IEEE Trans. on Systems, Man and Cybernetics, vol. SMC-13, no. 3, pp. 245-250, 1983.

[12] J.A. Cadzow, "Algorithm for the minimum-effort problem," in IEEE Transactions on Automatic Control,vol. AC-16, pp. 60-63 Feb. 1971.

[13] I-C. Shim, Y-S. Yoon, "Stabilized minimum infinity-norm torque solution for redundant manipulator," in Robotica, 16, pp. 193-205, 2001.

[14] I. Ha, J. Lee, "A new algorithm for a minimum infinity-norm solution and its application to trajectory planning of kinematically redundant manipulators," in IEEE/ASME Int. Conf. on Advanced Intelligent Mechatronics, vol. 1, pp. 446-451.

[15] Y. Zhang, "Inverse-free computation for infinity-norm torque minimization of robot manipulators," in Mechatronics, vol. 16 (3-4), pp. 177-184, 2006.

[16] I. A. Gravagne, I. D. Walker, "On the structure of minimum effort solutions with application to kinematic redundancy resolution," in IEEE Trans. on Robotics and Automation, vol. 16, pp. 855-863, 2000.

[17] V. Salvucci, Y. Kimura, O. Sehoon, and Y. Hori, "Force maximization of biarticularly actuated manipulators using infinity norm," in IEEE/ASME Transactions on Mechatronics, vol. 18, no. 3, pp. 1080-1089, 2013.

[18] A. Viehweider, V. Salvucci, Y. Hori, and T. Koseki, "Improving EV lateral dynamics control using infinity norm approach with closed-form solution," in Int. Conf. on Mechatronics, 2013.